

IRRATIONALITY OF MOTIVIC SERIES OF CHOW VARIETIES

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1. INTRODUCTION

The Euler characteristic of all the Chow varieties, of a fixed projective variety, can be collected in a formal power series called the Euler-Chow series. This series coincides with the Hilbert series when the Picard group is a finite generated free abelian group. It is an interesting open problem to find for which varieties this series is rational. A few cases have been computed, and it is suspected that the series is not rational for the blow up of \mathbb{P}^2 at nine points in general position.

It is very natural to extend this series to Chow motives and ask the question if the series is rational or to find a counterexample. In this short paper we generalized the series and show by an example that the series is not rational. This opens the question of what is the geometrical meaning of the Euler-Chow series.

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2. HOMOLOGICAL CHOW MOTIVES

In this section we recall some definitions that we need in the next sections.

Definition 2.1. *When X and Y are smooth complete varieties over \mathbb{C} , an element $\alpha \in \mathrm{CH}_*(X \times Y)$ is called a correspondence from X to*

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Y , and written as $\alpha : X \vdash Y$. When $\alpha : X \vdash Y$ and $\beta : Y \vdash Z$ are correspondences, we define their composition $\beta \circ \alpha : X \vdash Z$ by

$$\beta \circ \alpha = \pi_{XZ*}(\pi_{XY}^* \alpha \cdot \pi_{YZ}^* \beta)$$

where π are projections from $X \times Y \times Z$ to the products of indicated components, and $_ \cdot _$ denotes the intersection product.

Let $X = \coprod X_i$ be the decomposition of X into the irreducible (hence connected) components. Then via the canonical isomorphism $\mathrm{CH}_*(X \times Y) \simeq \oplus \mathrm{CH}_*(X_i \times Y)$, a correspondence $\alpha : X \vdash Y$ decomposes into $\alpha = \oplus \alpha_i$, with $\alpha_i \in \mathrm{CH}_*(X_i \times Y)$. A correspondence $\alpha : X \vdash Y$ is said to have relative dimension d when $\alpha_i \in \mathrm{CH}_{d+\dim X_i}(X_i \times Y)$.

Definition 2.2. A (homological) Chow motive is a triple (X, p, n) where X is a smooth complete scheme over \mathbb{C} , $p : X \vdash X$ is a correspondence with relative dimension 0 such that $p \circ p = p$ (namely p is idempotent), and n is an integer. The Chow motive $(X, [\Delta_X], 0)$ is called the **Chow motive of X** and it is written as $h(X)$, where $\Delta_X \subset X \times X$ is the diagonal subvariety.

When $M = (X, p, n)$ and $N = (Y, q, m)$ are Chow motives, then a **morphism of Chow motives** from M to N is a correspondence $\alpha : X \vdash Y$ such that α has a relative dimension $n - m$ and $q \circ \alpha \circ p = \alpha$. Finally, the category of Chow motives is denoted as ChMot .

When $f : X \rightarrow Y$ is a morphism of smooth complete schemes, then its graph $[\Gamma_f] \in \mathrm{CH}_*(X \times Y)$ determines a morphism $h(X) \rightarrow h(Y)$, and this morphism is written as $h(f)$.

For a motive $M = (X, p, n)$, we define its i -th homology $H_i(M)$ to be the image of the projector

$$H_i(M) := p_*(H_{i+2n}(X, \mathbb{Q})) \subset H_{i+2n}(X, \mathbb{Q}).$$

The tensor product $M \otimes N$ of two Chow motives $M = (X, p, n)$ and $N = (Y, q, m)$ is defined to be

$$M \otimes N := (X \times Y, \pi_X^* p \cdot \pi_Y^* q, n + m)$$

where $\pi_X : (X \times Y) \times (X \times Y) \rightarrow X \times X$ and $\pi_Y : (X \times Y) \times (X \times Y) \rightarrow Y \times Y$ are projections, and “ \cdot ” is the intersection product.

Remark 2.3. h determines a covariant functor from the category of smooth complete schemes to the category of Chow motives.

Remark 2.4. Classically, Chow motives are defined cohomologically, and the cohomology functor h is a contravariant functor (cf. [Sch94]). In this paper, we need homological operation on the Chow motives rather than cohomological one, so we choose to use homological (and hence covariant) definition of Chow motives.

Lemma 2.5. *Let $P \in \mathbb{P}^1$ be some point in \mathbb{P}^1 , and consider a point $Pt := \text{Spec}(\mathbb{C})$ outside of \mathbb{P}^1 . Then, in the category of Chow motives, we have an isomorphism*

$$(\mathbb{P}^1, [P \times \mathbb{P}^1], -1) \simeq (Pt, [\Delta_{Pt}], 0).$$

Furthermore, for any motive M there is an isomorphism

$$M \otimes (Pt, [\Delta_{Pt}], 0) \simeq M.$$

Proof. The morphisms $[P \times Pt] : \mathbb{P}^1 \vdash Pt$ and $[Pt \times \mathbb{P}^1] : Pt \vdash \mathbb{P}^1$ give the isomorphism between $(\mathbb{P}^1, [P \times \mathbb{P}^1], -1)$ and $(Pt, [\Delta_{Pt}], 0)$. The isomorphism $M \otimes (Pt, [\Delta_{Pt}], 0) \simeq M$ follows immediately from the definition of tensor product. \square

Corollary 2.6. *We have an isomorphism of Chow motives*

$$(X, p, n) \simeq (X \times (\mathbb{P}^1)^{n-m}, p \times [(P \times \mathbb{P}^1)^{n-m}], m),$$

where $P \in \mathbb{P}^1$ is any fixed point.

Definition 2.7. *Let $M = (X, p, n)$ and $N = (Y, q, m)$ be Chow motives. When $n = m$, we define $M \oplus N$ by*

$$M \oplus N := (X \amalg Y, p \amalg q, n).$$

When $n \neq m$, say $n > m$, we define $M \oplus N$ by

$$M \oplus N := (X \times (\mathbb{P}^1)^{n-m} \amalg Y, p \times [(P \times \mathbb{P}^1)^{n-m}] \amalg q, m).$$

Definition 2.8. *We define $K_0(\text{ChMot})$ to be $(\text{ChMot} \times \text{ChMot}) / \sim$ where $(M_1, N_1) \sim (M_2, N_2)$ iff $M_1 \oplus N_2 \simeq M_2 \oplus N_1$. We write the object (M, N) as $[M] - [N]$.*

Remark 2.9. Usual argument says that $K_0(\mathcal{ChMot})$ is a ring with \oplus as addition and \otimes as multiplication.

Remark 2.10. The cohomology series extends to $K_0(\mathcal{ChMot})$ by sending $[M] - [N]$ to $\sum (\dim H_i(M) - \dim H_i(N))t^i$. This gives a ring homomorphism from $K_0(\mathcal{ChMot}) \rightarrow \mathbb{Z}[t]$. By further substituting $t = -1$, there is a ring homomorphism $K_0(\mathcal{ChMot}) \rightarrow \mathbb{Z}$, which sends $h(X)$ to $\chi(X)$, the Euler characteristic of X .

3. CHOW SERIES OF CHOW VARIETIES

In this section we define the *Chow series* for Chow motives which generalizes the *Euler-Chow series* for Chow varieties defined in [Eli94] and [ELF98].

Definition 3.1. Let X be a smooth projective scheme, and $\lambda \in H_{2p}(X, \mathbb{Z})$ be its homology class. Define $\mathcal{C}_\lambda(X)$ to be the space of all effective cycles on X whose homology class is λ . We define $\mathcal{C}_p(X)$ to be the disjoint union of all $\mathcal{C}_\lambda(X)$ with $\lambda \in H_{2p}(X, \mathbb{Z})$.

Remark 3.2. The space $\mathcal{C}_\lambda(X)$ is an open and closed subscheme of the classical Chow variety $\mathcal{C}_{p,d}(X)$ where d is the degree of λ for the given projective structure, hence $\mathcal{C}_\lambda(X)$ is a projective scheme. See [Eli94, Lemma 1.1].

Remark 3.3. The space of all effective p -cycles $\mathcal{C}_p(X)$ has the canonical monoid structure, $(\alpha, \beta) \mapsto \alpha + \beta$, the addition of p -cycles, which is compatible with the addition of the homology class: $\mathcal{C}_\lambda(X) \times \mathcal{C}_\mu(X) \rightarrow \mathcal{C}_{\lambda+\mu}(X)$.

Definition 3.4. Let C be the monoid of homology classes of effective p -cycles in $H_{2p}(X, \mathbb{Z})$, and $K_0(\mathcal{ChMod})[[C]]$ be the set of all $K_0(\mathcal{ChMod})$ valued functions on C .

Remark 3.5. $K_0(\mathcal{ChMot})[[C]]$ has a ring structure with multiplication defined by the convolution product, i.e., for $f, g \in K_0(\mathcal{ChMot})[[C]]$, we define

$$(f \times g)(\lambda) := \sum_{\lambda = \mu_1 + \mu_2} f(\mu_1) \otimes f(\mu_2)$$

See [Eli94, Prop. 1.3] for the well-definedness of this product.

Definition 3.6. *Let X be a projective variety. We define the Chow series of X , in dimension p , to be the element $C_p(X) \in K_0(\mathcal{ChMot})[[C]]$ by $C_p(X)(\lambda) = [\mathcal{C}_\lambda(X)]$ for $\lambda \in H_{2p}(X, \mathbb{Z})$. By convention, if \mathcal{C}_λ is the empty set ϕ , then we define $[\phi] = 0$ in $K_0(\mathcal{ChMot})$.*

Definition 3.7. *Let $K_0(\mathcal{ChMot})[C]$ be the monoid-ring of C over $K_0(\mathcal{ChMot})$. This ring consists of all elements of $K_0(\mathcal{ChMot})[[C]]$ with finite support, and sometimes we refer to an element of $K_0(\mathcal{ChMot})[C]$ as a polynomial.*

Definition 3.8. *An element $\varphi \in K_0(\mathcal{ChMot})[[C]]$ is rational if for some polynomials $f, g \in K_0(\mathcal{ChMot})[C]$ with f invertible in $K_0(\mathcal{ChMot})[[C]] \otimes \mathbb{Q}$, we have $f\varphi = g$ in $K_0(\mathcal{ChMot})[[C]]$.*

Remark 3.9. It is not known if $K_0(\mathcal{ChMot})$ is an integral domain. A polynomial $f \in K_0(\mathcal{ChMot})[C]$ is invertible in $K_0(\mathcal{ChMot})[[C]] \otimes \mathbb{Q}$ iff $f(0)$ is invertible in $K_0(\mathcal{ChMot}) \otimes \mathbb{Q}$, where $0 \in H_{2p}(X, \mathbb{Z})$ is the zero object.

Remark 3.10. If we assume the finite dimensional conjecture (see [Kim05]), or the stronger assertion of the Bloch-Beilinson conjecture (see [Jan94]), then for $p = 0$, the Chow series $\mathcal{C}_0(X)$ is rational (see [And05]). Also by using the ring homomorphism from $K_0(\mathcal{ChMot}) \rightarrow \mathbb{Z}$ which sends $h(X)$ to $\chi(X)$ (see Remark 2.10), if we know that $\mathcal{C}_p(X)$ is rational, then it implies that the Euler-chow Series $E_p(X) = \sum_\lambda \chi(\mathcal{C}_\lambda) \lambda$ is also rational. The series is known to be rational for toric varieties (see [Eli94]), particular cases for Grassmannians (see [ELF98]) and some Del Pezzo surfaces.

Remark 3.11. The rational elements of $K_0(\mathcal{ChMot})[[C]]$ is closed under addition and multiplication. Also if a rational element is invertible, then its inverse is also rational.

4. CHOW SERIES OF PROJECTIVE SPACES

In this section, we prove that the Chow series $C_{n-1}(\mathbb{P}^n)$ is not rational, if $n > 1$.

Lemma 4.1. *Let $CS : K_0(\mathcal{ChMot})[[C]] \rightarrow \mathbb{Z}[t][[C]]$ be the cohomology series homomorphism. If $CS(\varphi)$ is not a rational function in $\mathbb{Q}[t][[C]]$ for $\varphi \in K_0(\mathcal{ChMot})[[C]]$, then φ is not rational.*

Proof. If φ is rational, then there are polynomials $f, g \in K_0(\mathcal{ChMot})[C]$ with f invertible in $K_0(\mathcal{ChMot})[[C]] \otimes \mathbb{Q}$ such that $f\varphi = g$. We send both sides by the ring homomorphism CS to find that $CS(f)CS(\varphi) = CS(g)$, where $CS(f)$ and $CS(g)$ are polynomials in $\mathbb{Z}[t][C]$. Let $h \in K_0(\mathcal{ChMot})[[C]] \otimes \mathbb{Q}$ be the inverse of f , then $CS(f)CS(g) = 1$ in $\mathbb{Q}[t][[C]]$. In particular, the constant term of $CS(f)$ is not zero, and hence non zero divisor, and we can write $CS(\varphi) = CS(g)/CS(f)$ as a rational function, a contradiction. \square

Definition 4.2. *Let $\varphi \in \mathbb{Z}[[s, t]]$ be a power series. We write φ as*

$$\varphi(s, t) = \sum_{i=0}^{\infty} a_i(t)s^i$$

where $a_i(t) \in \mathbb{Z}[[t]]$ for each i . We say that φ has a gap sequence $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$, with $a_i \in \mathbb{N} \ \forall i$, if there is a sequence of natural numbers $d_1 < d_2 < d_3 < \dots$ such that $a_{d_i}(t) \neq 0$ and $a_{d_i+1}(t) = a_{d_i+2}(t) = \dots = a_{d_i+\alpha_i}(t) = 0$.

Lemma 4.3. *If $\varphi \in \mathbb{Z}[[s, t]]$ has a gap sequence $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ such that $\lim_{n \rightarrow \infty} \alpha_n = \infty$, then φ is not a rational function. Namely, φ is not in $\mathbb{Q}(s, t)$.*

Proof. Assume that $\varphi(s, t)f(s, t) = g(s, t)$ for some polynomials $f(s, t)$ and $g(s, t)$ in $\mathbb{Z}[s, t]$. By the definition of the gap sequence, we can take $d_1 < d_2 < d_3 < \dots$. When i is large enough, we have $\alpha_i > \deg_s f$ and $d_i > \deg_s g$. Then term of $\varphi(s, t)f(s, t)$ with the degree in s to be $\deg_s f + d_i$ is non zero, and on the other hand, the term of $g(s, t)$ with the degree in s to be $\deg_s f + d_i$ is zero, a contradiction. \square

Proposition 4.4. *The cohomology series of $\mathcal{C}_{n-1}(\mathbb{P}^n)$ is given by*

$$\sum_i \dim H_i(\mathcal{C}_{n-1}(\mathbb{P}^n)) t^i = \sum \frac{1 - s^{2\binom{d+n}{d}}}{1 - s^2} t^d$$

Proof. $H_{2n-2}(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$, and one can identify $C = \{0, 1, 2, \dots\}$, by degree. The degree d part of the Chow variety is $\mathcal{C}_{n-1,d}(\mathbb{P}^n) \simeq \mathbb{P}^{\binom{d+n}{d}-1}$, from which the proposition immediately follows. \square

Theorem 4.5. *The Chow series $\sum [\mathcal{C}_{n-1,d}] t^d \in K_0(\mathcal{ChMot})[[C]]$ is not rational.*

Proof. By Lemma 4.1, it is enough to show that

$$\varphi := \sum \frac{1 - s^{2\binom{d+n}{d}}}{1 - s^2} t^d$$

is not rational in $\mathbb{Q}(s, t)$. Assume that φ is a rational function, then

$$(1 - s^2)\varphi = \sum \left(1 - s^{2\binom{d+n}{d}}\right) t^d$$

is also a rational function. But by setting $d_i := 2\binom{d+n}{d}$, $(1 - s^2)\varphi$ has the gap sequence $\{\alpha_1, \alpha_2, \dots\}$ such that

$$\alpha_d = 2 \binom{d+1+n}{d+1} - 2 \binom{d+n}{d} = 2 \binom{d+n}{n-1}$$

which tends to infinity. By Lemma 4.3, it implies that φ is not rational. \square

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